RESISTANCE OF A LIQUID TO THE MOTION OF A GAS BUBBLE COMPRESSED BY PARALLEL WALLS

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It is well known [1] that for Reynolds numbers of $N_{Re} < 1$ the hydrodynamic equations of an incompressible liquid incorporating the Stokes approximation agree closely with experiment in relation to the resistance coefficients of cylinders, spheres, and other bodies.

In this paper we shall apply the approximation in question to the motion of a gas bubble severely compressed by parallel plane walls (Fig. 1). The shape of the bubble is taken as almost cylindrical in view of the fact that, when the bubble is severely crushed, the difference $(r_0 - r_1)$ is much smaller than r_1 (the distance from the axis of the bubble to the wetting point) and r_0 (the distance from the axis to the meniscus). We may therefore approximately regard the bubble as cylindrical with a radius r_0.

If in the Navier-Stokes equations we reject the inertial terms, the equations for the steady-state case will take the form

$$1 / \rho \operatorname{grad} p = \nu \Delta \mathbf{V} \quad (1)$$

In addition to this we have the continuity equation

$$\operatorname{div} \mathbf{V} = 0$$

Applying the operation div to both sides of (1), we obtain

$$\operatorname{div} \operatorname{grad} p = 0 \quad (2)$$

The system of equations (1) and (2) determines the flow field in the Stokes approximation.

Let us introduce a rectangular coordinate system with its x axis along the flow, its z axis perpendicular to the plane of the plate, and its y axis parallel to the plates.

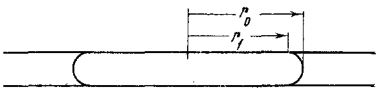

Fig. 1

A flow of viscous liquid approaches the cylinder from infinity (on the left) in the direction of the x axis. We shall consider that the bubble remains stationary, i.e., its resistance is compensated by some external force.

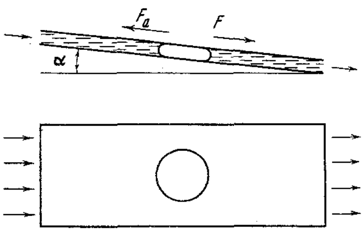

Fig. 2

We have to solve the system of equations (1) and (2) subject to the following conditions. On moving away from the bubble to infinity the flow degenerates into the well-known Poisseuille flow; the velocity component normal to the surface of the bubble is equal to zero at that surface. We assume that the force of surface tension is so great that the sections cut off from the bubble by planes parallel to the walls are almost circular.

Let us take as a linear scale the radius of the bubble r_0 and as a scale of velocity the velocity V_0 at infinity. For $\xi = x/r_0 \to \infty$ or $\eta = y/r_0 \to \infty$ only one velocity component will be different from zero

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$$\omega = V/V_0\,(u, v, o)$$

$$u = 1 - \frac{r_0^2}{h^2}\zeta^2 \qquad \left(\zeta = \frac{z}{r_0}\right) \tag{3}$$

Here 2h is the distance between the plates.

Let us introduce the polar coordinates R and φ

$$\xi = r\cos\varphi, \quad \eta = r\sin\varphi \qquad (r = R/r_0)$$

In these coordinates, Eq. (2) takes the form

$$\frac{\partial^2\pi}{\partial r^2} + \frac{1}{r}\frac{\partial\pi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\pi}{\partial\varphi^2} = 0 \qquad \left(\pi = \frac{p}{\rho V_0^2}\right) \tag{4}$$

Here π is the dimensionless pressure. We shall solve Eq. (4) by the method of variable separation and confine attention to the first term of the resultant series

$$\pi = a\,(r + b/r)\cos\varphi \tag{5}$$

where a and b are constants of integration which have to be determined. It is clear from the geometry of the flow and Eq. (5) that $\partial\pi/\partial\xi = a$ as $r \to \infty$. We have hitherto everywhere assumed that the pressure is independent of ζ and that the component of velocity in the ζ direction is zero. The components u and v depend on ζ, and we derive the following equations for these from (1):

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial\zeta^2} = aN_{Re}\left(1 - \frac{b}{r^2}\cos 2\varphi\right) \tag{6}$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial z} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\varphi^2} + \frac{\partial^2 v}{\partial\zeta^2} = -aN_{Re}\frac{b}{r^2}\sin 2\varphi \tag{7}$$

The values of u and v should vanish at $\zeta = \pm h/r_0$. Furthermore, the radial velocity $u_r = u\cos\varphi + v\sin\varphi$ at $r = 1$ should also vanish. By considering (3) and (6) as $r \to \infty$ we obtain the relation $aN_{Re} = -2r_0^2/h^2$.

Remembering the foregoing relationship, from (6) and (7) we obtain the solutions

$$u = \left(1 - \frac{r_0^2}{h^2}\zeta^2\right)\left(1 - \frac{b}{r^2}\right)\cos 2\varphi, \qquad v = -\left(1 - \frac{r_0^2}{h^2}\zeta^2\right)\frac{b}{r^2}\sin 2\varphi$$

From the two latter expressions we have for u_r

$$u_r = \left(1 - \frac{r_0^2}{h^2}\zeta^2\right)\left(1 - \frac{b}{r^2}\right)\cos\varphi$$

It follows from the foregoing and also from the conditions on the surface of the cylinder that b = 1 and hence

$$u_r = \left(1 - \frac{r_0^2}{h^2}\zeta^2\right)\left(1 - \frac{1}{r^2}\right)\cos\varphi \tag{8}$$

The surface of the cylinder has a "liquid" boundary, so that the tangential stress on the surface of the cylinder should be equal to zero. The solution here obtained only satisfies this condition approximately. However, in calculating the total force acting in the flow on the cylinder, we shall consider the tangential stress as being exactly zero, and the force to be determined will be the total force arising from all the stresses normal to the surface.

The normal force acting on an element of the cylinder [2] is

$$F_p = -p + 2\mu\frac{\partial U}{\partial R} \quad (U = uV_0) \tag{9}$$

where U is the dimensional component of the velocity V. Transforming to dimensional quantities in (8)

$$U_R = u_r V_0 = V_0\left(1 - \frac{z^2}{h^2}\right)\left(1 - \frac{r_0^2}{R^2}\right)\cos\varphi \tag{10}$$

From (10) we have

$$\left.\frac{\partial U_R}{\partial R}\right|_{R=r_0} = V_0\left(1-\frac{z^2}{h^2}\right)\frac{2}{r_0}\cos\varphi$$

The force acting on the cylinder

$$F = \int\limits_S F_R \cos\varphi\, r_0 dz d\varphi = 8\pi\mu V_0 h\left(\frac{2}{3}+\frac{r_0^2}{h^2}\right) \tag{11}$$

Under the assumptions made, the expression for the resistive force takes the form

$$F = 8\pi\mu V_0 r_0^2/h \qquad (r_0^2/h_0^2 \gg 1) \tag{12}$$

Equation (12) was tested experimentally. For this purpose we made an apparatus the working part of which (Fig. 2) was a plane channel with parallel walls separated by 2h = 1 mm; the length of the channel was L = 350 mm and the width D = 80 mm. The height of the channel 2h was chosen from the following considerations. The fall in hydrostatic pressure over the height of the channel should be smaller than the pressure inside the bubble for a bubble diameter of the order of 10 mm. For σ = 70 dyn/cm a value of the order of 1 mm is thus obtained for the height of the channel.

The external force holding the bubble stationary in the flow of liquid was the Archimedes force. To prevent the bubble from being distorted under the influence of the resistance of the liquid and the Archimedes force, and also to ensure that $N_{Re} < 1$, the slope of the channel (angle α) to the horizontal was made no greater than 5°. In the inclined position of the channel, the bubble was prevented from surfacing by suitably choosing the flow of liquid passing through the channel cross section.

Let the radius of the bubble be r_0, the slope of the channel plane to the horizontal α, the density of the liquid ρ, and the free-fall acceleration g. Then the Archimedes force of repulsion acting on the bubble may be expressed by the equation

$$F = 2\pi r_0^2 h\rho g \sin\alpha \tag{13}$$

Since the velocity profile a long way from the bubble has the form $V = V_0(1 - Z^2/h^2)$, the flow of liquid in unit time for a channel width of D is given by the expression

$$Q = DV_0 \int\limits_{-h}^{h}\left(1-\frac{z^2}{h^2}\right)dz = \frac{4}{3}DV_0 h \tag{14}$$

Using (12)-(14) we obtain

$$K = \frac{3Q\mu}{h^3 Dg\rho \sin\alpha} = 1 \tag{15}$$

Equation (15) is the equilibrium condition for the bubble in the channel. This condition does not depend on r_0; it thus follows that if a bubble with one particular radius is balanced experimentally, then a bubble of any other radius (subject to the aforementioned stipulations) will also be in equilibrium.

Let us give the experimental values of the complex K for various values of the slope (sin α) and bubble diameter ($2r_0$) in mm:

$\sin\alpha = 0.19$	0.3	0.35	0.37	0.52	0.63
$2r_0 = 6.0$	6.0	5.5	6.0	6.0	5.8
$K = 1.20$	0.89	0.81	1.14	1.10	1.09
$\sin\alpha = 0.63$	0.75	0.75	0.75	0.77	0.77
$2r_0 = 6.7$	5.0	8.8	10.2	5.5	9.2
$K = 1.29$	1.02	1.16	1.28	0.81	0.92

We see that K = 1.0 ± 0.15; furthermore, for a fixed value of α, K depends slightly on r_0; however, this dependence is not very severe (≈20%) and hardly exceeds the scatter of the experimental points.

We see from (12) that F is inversely proportional to h; this apparent contradiction is removed if we remember that Eq. (12) was obtained for the condition $r_0^2/h^2 \gg 1$, or if we consider Eq. (11). On increasing h and keeping the other parameters constant, not only does the area of the bubble alter, which should lead to an increase in the resistance, but the gradients in the velocity profile change also. On increasing h the velocity gradients diminish, which reduces the resistance to the motion. The resultant effect is expressed in Eqs. (11) and (12).

LITERATURE CITED

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2. D. Meksin, New Methods in Laminar Boundary Layer Theory, Pergamon Press, Oxford (1961).